DAY THREE

Sequence and Series

Learning & Revision for the Day

- Definition
- Arithmetic Progression (AP)
- Arithmetic Mean (AM)
- Geometric Progression (GP)
- Geometric Mean (GM)
- Arithmetico-Geometric Progression (AGP)
- Sum of Special Series
- Summation of Series by the Difference Method

Definition

- By a **sequence** we mean a list of numbers, arranged according to some definite rule.

  or

  We define a sequence as a function whose domain is the set of natural numbers or some subsets of type {1, 2, 3, ... k}.
- If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called the **series**.
- If the terms of a sequence follow a certain pattern, then it is called a **progression**.

Arithmetic Progression (AP)

- It is a sequence in which the difference between any two consecutive terms is always same.
- An AP can be represented as $a, a + d, a + 2d, a + 3d, \ldots$ where, a is the first term, d is the common difference.
- The nth term, $t_n = a + (n - 1)d$
- Common difference $d = t_n - t_{n-1}$
- The nth term from end, $t_n = l - (n - 1)d$, where l is the last term.
- Sum of first n terms, $S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + l]$, where l is the last term.
- If sum of n terms is S_n, then nth term is $t_n = S_n - S_{n-1}$, $t_n = \frac{1}{2}[t_{n-k} + t_{n+k}]$, where $k < n$

NOTE
- Any three numbers in AP can be taken as $a-d, a, a+d$.
- Any four numbers in AP can be taken as $a-3d, a-d, a+d, a+3d$.
- Any five numbers in AP can be taken as $a-2d, a-d, a, a+d, a+2d$.
- Three numbers a, b, c are in AP iff $2b = a + c$.

An Important Result of AP

- In a finite AP, $a_1, \ldots, a_n$, the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last term
  i.e. $a_1 + a_n = a_k + a_{n-(k-1)}, \forall k = 1, 2, 3, \ldots, n-1$.

Arithmetic Mean (AM)

- If a, A and b are in AP, then $A = \frac{a+b}{2}$ is the arithmetic mean of a and b.
- If $a, A_1, A_2, \ldots, A_n, b$ are in AP, then $A_1, A_2, \ldots, A_n$ are the n arithmetic means between a and b.
- The n arithmetic means, $A_1, A_2, \ldots, A_n$, between a and b are given by the formula, $A_r = a + \frac{r(b-a)}{n+1}\ \forall r = 1, 2, \ldots n$
- Sum of n AM's inserted between a and b is nA i.e.
  $A_1 + A_2 + A_3 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$

NOTE
- The AM of n numbers $a_1, a_2, \ldots, a_n$ is given by
  $\text{AM} = \frac{(a_1 + a_2 + a_3 + \ldots + a_n)}{n}$

Geometric Progression (GP)

- It is a sequence in which the ratio of any two consecutive terms is always same.
- A GP can be represented as $a, ar, ar^2, \ldots$
  where, a is the first term and r is the common ratio.
- The nth term, $t_n = ar^{n-1}$
- The nth term from end, $t_n' = \frac{l}{r^{n-1}}$, where l is the last term.
- Sum of first n terms, $S_n = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & r \neq 1 \\ na, & r = 1 \end{cases}$
- If $|r| < 1$, then the sum of infinite GP is $S_\infty = \frac{a}{1-r}$

NOTE
- Any three numbers in GP can be taken as $\frac{a}{r}, a, ar$.
- Any four numbers in GP can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
- Any five numbers in GP can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.
- Three non-zero numbers a, b, c are in GP iff $b^2 = ac$.
- If a, b and c are in AP as well as GP, then $a = b = c$.
- If $a > 0$ and $r > 1$ or $a < 0$ and $0 < r < 1$, then the GP will be an increasing GP.
- If $a > 0$ and $0 < r < 1$ or $a < 0$ and $r > 1$, then the GP will be a decreasing GP.

Important Results on GP

- If $a_1, a_2, a_3, \ldots, a_n$ is a GP of positive terms, then $\log a_1, \log a_2, \ldots, \log a_n$ is an AP and *vice-versa*.
- In a finite GP, $a_1, a_2, \ldots, a_n$, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.
  i.e. $a_1 a_n = a_k \cdot a_{n-(k-1)}, \forall k = 1, 2, 3, \ldots, n-1$.

Geometric Mean (GM)

- If a, G and b are in GP, then $G = \sqrt{ab}$ is the geometric mean of a and b.
- If $a, G_1, G_2, \ldots, G_n, b$ are in GP, then $G_1, G_2, \ldots, G_n$ are the n geometric means between a and b.
- The n GM's, $G_1, G_2, \ldots, G_n$, inserted between a and b, are given by the formula, $G_r = a\left(\frac{b}{a}\right)^{\frac{r}{n+1}}$.
- Product of n GM's, inserted between a and b, is the nth power of the single GM between a and b,
  i.e. $G_1 \cdot G_2 \cdot \ldots \cdot G_n = G^n = (ab)^{n/2}$.

NOTE
- If a and b are of opposite signs, then their GM can not exist.
- If A and G are respectively the AM and GM between two numbers a and b, then a, b are given by $[A \pm \sqrt{(A+G)(A-G)}]$.
- If $a_1, a_2, a_3, \ldots, a_n$ are positive numbers, then their GM $= (a_1\ a_2\ a_3 \ldots a_n)^{1/n}$.

Arithmetico-Geometric Progression (AGP)

- A progression in which every term is a product of a term of AP and corresponding term of GP, is known as arithmetico-geometric progression.
- If the series of AGP be $a + (a+d)r + (a+2d)r^2 + \ldots + \{a + (n-1)d\}r^{n-1} + \ldots$, then
  (i) $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{1-r}, r \neq 1$
  (ii) $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}, |r| < 1$

Method to find the Sum of n-terms of Arithmetic Geometric Progression

Usually, we do not use the above formula to find the sum of n terms.

Infact we use the mechanism by which we derived the formula, shown below:

Let, $S_n = a + (a + d)r + (a + 2d)r^2$

$$+ \dots + (a + (n-1)d)r^{n-1} \quad \dots\text{(i)}$$

Step I Multiply each term by r (Common ratio of GP) and obtain a new series

$\Rightarrow \quad rS_n = ar + (a+d)r^2 + \dots +$

$$(a + (n-2)d)r^{n-1} + (a + (n-1)d)r^n \quad \dots\text{(ii)}$$

Step II Subtract the new series from the original series by shifting the terms of new series by one term

$\Rightarrow (1-r)S_n = a + [dr + dr^2 + \dots + dr^{n-1}] - (a + (n-1)d)r^n$

$\Rightarrow \quad S_n(1-r) = a + dr\left(\dfrac{1-r^{n-1}}{1-r}\right) - (a + (n-1)d)r^n$

$\Rightarrow \quad S_n = \dfrac{a}{1-r} + dr\left(\dfrac{1-r^{n-1}}{(1-r)^2}\right) - \dfrac{(a + (n-1)d)}{1-r} r^n$

Sum of Special Series

- Sum of first n natural numbers,

$$1 + 2 + \dots + n = \Sigma n = \frac{n(n+1)}{2}$$

- Sum of squares of first n natural numbers,

$$1^2 + 2^2 + \dots + n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of first n natural numbers,

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(i) Sum of first n even natural numbers

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

(ii) Sum of first n odd natural numbers

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

Summation of Series by the Difference Method

If nth term of a series cannot be determined by the methods discussed so far. Then, nth term can be determined by the method of difference, if the difference between successive terms of series are either in AP or in GP, as shown below:

Let $T_1 + T_2 + T_3 + \dots$ be a given infinite series.

If $T_2 - T_1, T_3 - T_2, \dots$ are in AP or GP, then T_n can be found by following procedure.

Clearly, $S_n = T_1 + T_2 + T_3 + \dots + T_n \quad \dots\text{(i)}$

Again, $S_n = T_1 + T_2 + \dots + T_{n-1} + T_n \quad \dots\text{(ii)}$

$\therefore \quad S_n - S_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \dots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \dots + (T_n - T_{n-1})$

$\Rightarrow \quad T_n = T_1 + t_1 + t_2 + t_3 + t_{n-1}$

where, $t_1, t_2, t_3, \dots$ are terms of the new series $\Rightarrow S_n = \sum_{r=1}^{n} T_r$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $\log_3 2, \log_3 (2^x - 5)$ and $\log_3 \left(2^x - \dfrac{7}{2}\right)$ are in AP, then x is equal to

(a) 2 (b) 3 (c) 4 (d) 2, 3

2 The number of numbers lying between 100 and 500 that are divisible by 7 but not by 21 is

(a) 57 (b) 19 (c) 38 (d) None of these

3 If 100 times the 100th term of an AP with non-zero common difference equals the 50 times its 50th term, then the 150th term of this AP is

(a) – 150 (b) 150 times its 50th term
(c) 150 (d) zero

4 If $a_1, a_2, \dots, a_{n+1}$ are in AP, then $\dfrac{1}{a_1 a_2} + \dfrac{1}{a_2 a_3} + \dots + \dfrac{1}{a_n a_{n+1}}$ is

(a) $\dfrac{n-1}{a_1 a_{n+1}}$ (b) $\dfrac{1}{a_1 a_{n+1}}$ (c) $\dfrac{n+1}{a_1 a_{n+1}}$ (d) $\dfrac{n}{a_1 a_{n+1}}$

5 A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which are in AP. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. The value of the 8th instalment is

(a) ₹ 35 (b) ₹ 50
(c) ₹ 65 (d) None of these

6 Let $a_1, a_2, a_3, \dots$ be an AP, such that $\dfrac{a_1 + a_2 + \dots + a_p}{a_1 + a_2 + a_3 + \dots + a_q} = \dfrac{p^3}{q^3}$; $p \neq q$, then $\dfrac{a_6}{a_{21}}$ is equal to

→ **JEE Mains 2013**

(a) $\dfrac{41}{11}$ (b) $\dfrac{121}{1681}$
(c) $\dfrac{11}{41}$ (d) $\dfrac{121}{1861}$

7 A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the nth minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference – 2, then the time taken by him to count all notes, is

(a) 24 min (b) 34 min (c) 125 min (d) 135 min

8 If $\log_{\sqrt{3}} a^2 + \log_{(3)^{1/3}} a^2 + \log_{3^{1/4}} a^2 + \ldots$ upto 8th term $= 44$, then the value of a is

(a) $\pm\sqrt{3}$ (b) $2\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) None of these

9 n arithmetic means are inserted between 7 and 49 and their sum is found to be 364, then n is

(a) 11 (b) 12 (c) 13 (d) 14

10 If $x = 111\ldots1$ (20 digits), $y = 333\ldots3$ (10 digits) and $z = 222\ldots2$ (10 digits), then $\frac{x - y^2}{z}$ is equal to

(a) 1 (b) 2 (c) 1/2 (d) 3

11 If the 2nd, 5th and 9th terms of a non-constant AP are in GP, then the common ratio of this GP is ➜ **JEE Mains 2016**

(a) $\frac{8}{5}$ (b) $\frac{4}{3}$ (c) 1 (d) $\frac{7}{4}$

12 Three positive numbers form an increasing GP. If the middle term in this GP is doubled,then new numbers are in AP. Then, the common ratio of the GP is ➜ **JEE Mains 2014**

(a) $\sqrt{2}+\sqrt{3}$ (b) $3+\sqrt{2}$ (c) $2-\sqrt{3}$ (d) $2+\sqrt{3}$

13 A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places,then its common ratio is

(a) 2 (b) 3 (c) 4 (d) 5

14 The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, …, is ➜ **JEE Mains 2013**

(a) $\frac{7}{81}[179 - 10^{20}]$ (b) $\frac{7}{9}[99 - 10^{-20}]$

(c) $\frac{7}{81}[179 + 10^{-20}]$ (d) $\frac{7}{9}[99 + 10^{-20}]$

15 If x, y and z are distinct prime numbers, then

(a) x, y and z may be in AP but not in GP
(b) x, y and z may be in GP but not in AP
(c) x, y and z can neither be in AP nor in GP
(d) None of the above

16 Let $n\,(>1)$ be a positive integer, then the largest integer m such that $(n^m + 1)$ divides $(1 + n + n^2 + \ldots + n^{127})$, is

(a) 32 (b) 8 (c) 64 (d) 16

17 An infinite GP has first term x and sum 5, then x belongs

(a) $x < -10$ (b) $-10 < x < 0$ (c) $0 < x < 10$ (d) $x > 10$

18 The length of a side of a square is a metre. A second square is formed by joining the mid-points of these squares. Then, a third square is formed by joining the mid-points of the second square and so on. Then, sum of the area of the squares which carried upto infinity is

(a) a^2m^2 (b) $2a^2m^2$ (c) $3a^2m^2$ (d) $4a^2m^2$

19 If $|a| < 1$ and $|b| < 1$, then the sum of the series $1 + (1+a)b + (1+a+a^2)b^2 + (1+a+a^2+a^3)b^3 + \ldots$ is

(a) $\frac{1}{(1-a)(1-b)}$ (b) $\frac{1}{(1-a)(1-ab)}$

(c) $\frac{1}{(1-b)(1-ab)}$ (d) $\frac{1}{(1-a)(1-b)(1-ab)}$

20 A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹ 40 more than the saving of immediately previous month. His total saving from the start of service will be ₹ 11040 after ➜ **AIEEE 2011**

(a) 19 months (b) 20 months
(c) 21 months (d) 18 months

21 If one GM, g and two AM's, p and q are inserted between two numbers a and b, then $(2p - q)(p - 2q)$ is equal to

(a) g^2 (b) $-g^2$ (c) $2g$ (d) $3g^2$

22 If five GM's are inserted between 486 and $\frac{2}{3}$, then fourth GM will be

(a) 4 (b) 6 (c) 12 (d) – 6

23 The sum to 50 terms of the series

$$1 + 2\left(1 + \frac{1}{50}\right) + 3\left(1 + \frac{1}{50}\right)^2 + \ldots, \text{ is given by}$$

(a) 2500 (b) 2550
(c) 2450 (d) None of these

24 If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to ➜ **JEE Mains 2014**

(a) $\frac{121}{10}$ (b) $\frac{441}{100}$ (c) 100 (d) 110

25 The sum of the infinity of the series

$$1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots \text{ is}$$

(a) 3 (b) 4 (c) 6 (d) 2

26 The sum of the series $1^3 + 3^3 + 5^3 + \ldots$ upto 20 terms is

(a) 319600 (b) 321760
(c) 306000 (d) 347500

27 Let $a_1, a_2, a_3, \ldots, a_{49}$ be in AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots + a_{17}^2 = 140\,m$, then m is equal to ➜ **JEE Mains 2018**

(a) 66 (b) 68 (c) 34 (d) 33

28 Let $a, b, c \in R$. if $f(x) = ax^2 + bx + c$ be such that $a + b + c = 3$ and $f(x+y) = f(x) + f(y) + xy$, $\forall\, x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to ➜ **JEE Mains 2017**

(a) 330 (b) 165 (c) 190 (d) 255

29 The sum of the series $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$ upto 10 terms is ➜ **JEE Mains 2013**

(a) 11300 (b) 11200 (c) 12100 (d) 12300

30 Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \ldots$

If $B - 2A = 100\lambda$, then λ is equal to ➜ **JEE Mains 2018**

(a) 232 (b) 248 (c) 464 (d) 496

31 The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$ is ➜ **JEE Mains 2015**

(a) 71 (b) 96 (c) 142 (d) 192

32 The sum $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \ldots$ upto 11 terms is ➜ **JEE Mains 2013**

(a) $\frac{7}{2}$ (b) $\frac{11}{4}$ (c) $\frac{11}{2}$ (d) $\frac{60}{11}$

33 The sum of the series $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ upto 10 terms is ➜ **JEE Mains 2013**

(a) $\frac{18}{11}$ (b) $\frac{22}{13}$ (c) $\frac{20}{11}$ (d) $\frac{16}{9}$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 The value of $1^2 + 3^2 + 5^2 + \ldots + 25^2$ is ➜ **JEE Mains 2013**

(a) 2925 (b) 1469 (c) 1728 (d) 1456

2 If the function f satisfies the relation $f(x + y) = f(x)\cdot f(y)$ for all natural numbers $x, y, f(1) = 2$ and $\sum_{r=1}^{n} f(a + r) = 16(2^n - 1)$, then the natural number a is

(a) 2 (b) 3 (c) 4 (d) 5

3 If the sum of an infinite GP is $\frac{7}{2}$ and sum of the squares of its terms is $\frac{147}{16}$, then the sum of the cubes of its terms is

(a) $\frac{315}{19}$ (b) $\frac{700}{39}$ (c) $\frac{985}{13}$ (d) $\frac{1029}{38}$

4 The sum of the infinite series $\frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \ldots$ is

(a) $\frac{31}{18}$ (b) $\frac{65}{32}$ (c) $\frac{65}{36}$ (d) $\frac{75}{36}$

5 Given sum of the first n terms of an AP is $2n + 3n^2$. Another AP is formed with the same first term and double of the common difference, the sum of n terms of the new AP is

(a) $n + 4n^2$ (b) $6n^2 - n$ (c) $n^2 + 4n$ (d) $3n + 2n^2$

6 For $0 < \theta < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta$, then xyz is equal to

(a) $xz + y$ (b) $x + y + z$ (c) $yz + x$ (d) $x + y - z$

7 If $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$ to $\infty = \frac{\pi^4}{90}$, then $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ to ∞ is equal to

(a) $\frac{\pi^4}{96}$ (b) $\frac{\pi^4}{45}$ (c) $\frac{89\pi^4}{90}$ (d) $\frac{\pi^4}{90}$

8 If S_n is the sum of first n terms of a GP : $\{a_n\}$ and S_n' is the sum of another GP : $\{1/a_n\}$, then S_n equals

(a) $\frac{S_n'}{a_1 a_n}$ (b) $a_1 a_n S_n'$ (c) $\frac{a_1}{a_n} S_n'$ (d) $\frac{a_n}{a_1} S_n'$

9 If the sum of the first ten terms of the series

$\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$, is $\frac{16}{5}m$,

then m is equal to ➜ **JEE Mains 2016**

(a) 102 (b) 101 (c) 100 (d) 99

10 If m is the AM of two distinct real numbers l and n $(l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals ➜ **JEE Mains 2015**

(a) $4l^2mn$ (b) $4\,lm^2n$
(c) $4\,lmn^2$ (d) $4l^2m^2n^2$

11 The sum of the series $(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \ldots \infty$ is

(a) $\sqrt{2}$ (b) $2 + 3\sqrt{2}$ (c) $2 - 3\sqrt{2}$ (d) $\frac{4 + 3\sqrt{2}}{2}$

12 The largest term common to the sequences 1, 11, 21, 31, ... to 100 terms and 31, 36, 41, 46, ... to 100 terms is

(a) 531 (b) 471 (c) 281 (d) 521

13 If a, b, c are in GP and x is the AM between a and b, y the AM between b and c, then

(a) $\frac{a}{x} + \frac{c}{y} = 1$ (b) $\frac{a}{x} + \frac{c}{y} = 2$
(c) $\frac{a}{x} + \frac{c}{y} = 3$ (d) None of these

14 Suppose a, b and c are in AP and a^2, b^2 and c^2 are in GP. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

15 For any three positive real numbers a, b and c, if

$9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, then

(a) b, c and a are in GP ➜ **JEE Mains 2017**
(b) b, c and a are in AP
(c) a, b and c are in AP
(d) a, b and c are in GP

16 If $S_1, S_2, S_3, \ldots$ are the sum of infinite geometric series whose first terms are 1, 2, 3, ... and whose common ratios $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ respectively, then

$S_1^2 + S_2^2 + S_3^2 + \ldots + S_{10}^2$ is equal to

(a) 485 (b) 495
(c) 500 (d) 505

17 **Statement I** The sum of the series

$1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$ is 8000.

Statement II $\sum_{k=1}^{n} [k^3 - (k-1)^3] = n^3$, for any natural number n.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is ture; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1	**1** (b)	**2** (c)	**3** (d)	**4** (d)	**5** (c)	**6** (b)	**7** (b)	**8** (a)	**9** (c)	**10** (a)
	11 (b)	**12** (d)	**13** (c)	**14** (c)	**15** (a)	**16** (c)	**17** (c)	**18** (b)	**19** (c)	**20** (c)
	21 (b)	**22** (b)	**23** (a)	**24** (c)	**25** (a)	**26** (a)	**27** (c)	**28** (a)	**29** (c)	**30** (b)
	31 (b)	**32** (c)	**33** (c)							
SESSION 2	**1** (a)	**2** (b)	**3** (d)	**4** (c)	**5** (b)	**6** (b)	**7** (a)	**8** (b)	**9** (b)	**10** (b)
	11 (d)	**12** (d)	**13** (b)	**14** (d)	**15** (b)	**16** (d)	**17** (a)			

Hints and Explanations

SESSION 1

1 $\because 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \quad (2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \quad t^2 + 25 - 10t = 2t - 7$ [put $2^x = t$]

$\Rightarrow \quad t^2 - 12t + 32 = 0$

$\Rightarrow \quad (t-8)(t-4) = 0$

$\Rightarrow \quad 2^x = 8$ or $2^x = 4$

$\therefore \quad x = 3$ or $x = 2$

At, $x = 2$, $\log_3(2^x - 5)$ is not defined.

Hence, $x = 3$ is the only solution.

2 The numbers between 100 and 500 that are divisible by 7 are 105, 112, 119, 126, ..., 490, 497.

Let such numbers be n.

$\therefore \quad t_n = a_n + (n-1)d$

$\Rightarrow \quad 497 = 105 + (n-1) \times 7$

$\Rightarrow \quad n - 1 = 56$

$\Rightarrow \quad n = 57$

The numbers between 100 and 500 that are divisible by 21 are 105, 126, 147, ..., 483.

Let such numbers be m.

$\therefore \quad 483 = 105 + (m-1) \times 21$

$\Rightarrow \quad 18 = m - 1 \Rightarrow m = 19$

$\therefore$ Required number

$= n - m = 57 - 19 = 38$

3 Let a be the first term and d $(d \neq 0)$ be the common difference of a given AP, then

$T_{100} = a + (100-1)d = a + 99d$

$T_{50} = a + (50-1)d = a + 49d$

$T_{150} = a + (150-1)d = a + 149d$

Now, according to the given condition,

$100 \times T_{100} = 50 \times T_{50}$

$\Rightarrow \quad 100(a + 99d) = 50(a + 49d)$

$\Rightarrow \quad 2(a + 99d) = (a + 49d)$

$\Rightarrow \quad 2a + 198d = a + 49d$

$\Rightarrow \quad a + 149d = 0$

$\therefore \quad T_{150} = 0$

4 Let d be the common difference of given AP and let

$S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_na_{n+1}}$. Then,

$S = \frac{1}{d}\left[\frac{d}{a_1a_2} + \frac{d}{a_2a_3} + \ldots + \frac{d}{a_na_{n+1}}\right]$

$= \frac{1}{d}\left[\frac{a_2 - a_1}{a_1a_2} + \frac{a_3 - a_2}{a_2a_3} + \ldots + \frac{a_{n+1} - a_n}{a_na_{n+1}}\right]$

$= \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \ldots + \left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)\right]$

$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_{n+1}}\right] = \frac{1}{d}\left[\frac{a_{n+1} - a_1}{a_1a_{n+1}}\right]$

$= \frac{1}{d}\left[\frac{a_1 + nd - a_1}{a_1a_{n+1}}\right] = \frac{n}{a_1a_{n+1}}$

5 Given, $3600 = \frac{40}{2}[2a + (40-1)d]$

$\Rightarrow \quad 3600 = 20(2a + 39d)$

$\Rightarrow \quad 180 = 2a + 39d \quad \ldots(i)$

After 30 instalments one-third of the debt is unpaid.

Hence, $\frac{3600}{3} = 1200$ is unpaid and 2400 is paid.

Now, $\quad 2400 = \frac{30}{2}\{2a + (30-1)d\}$

$\therefore \quad 160 = 2a + 29d \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$a = 51, d = 2$

Now, the value of 8th instalment

$= a + (8-1)d$

$= 51 + 7 \cdot 2 = ₹\, 65$

6 Given that, $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^3}{q^3}$

$\Rightarrow \quad \frac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_2 + (q-1)d]} = \frac{p^3}{q^3}$

where, d is a common difference of an AP.

$\Rightarrow \quad \frac{2a_1 + (p-1)d}{2a_2 + (q-1)d} = \frac{p^2}{q^2}$

$\Rightarrow \quad \frac{a_1 + (p-1)\frac{d}{2}}{a_2 + (q-1)\frac{d}{2}} = \frac{p^2}{q^2}$

On putting $p = 11$ and $q = 41$, we get

$\frac{a_1 + (11-1)\frac{d}{2}}{a_2 + (41-1)\frac{d}{2}} = \frac{(11)^2}{(41)^2}$

$\Rightarrow \quad \frac{a_1 + 5d}{a_2 + 20d} = \frac{121}{1681}$

$\Rightarrow \quad \frac{a_6}{a_{21}} = \frac{121}{1681}$

7 Number of notes that the person counts in 10 min

$= 10 \times 150 = 1500$

Since, $a_{10}, a_{11}, a_{12}, \ldots$ are in AP with common difference -2.

Let n be the time taken to count remaining 3000 notes.

Then, $\frac{n}{2}[2 \times 148 + (n-1) \times -2] = 3000$

$\Rightarrow \quad n^2 - 149n + 3000 = 0$

$\Rightarrow \quad (n-24)(n-125) = 0$

$\therefore \quad n = 24$ and 125

Then, the total time taken by the person to count all notes

$= 10 + 24$

$= 34$ min

8 $S_n = \log a^2 \left[\frac{1}{\frac{1}{2}\log 3} + \frac{1}{\frac{1}{3}\log 3} + \frac{1}{\frac{1}{4}\log 3} + \ldots \text{ upto 8th term}\right]$

$\Rightarrow \quad \frac{\log a^2}{\log 3}[2 + 3 + 4 + \ldots + 9] = 44$ [given]

$\Rightarrow \quad 44 \log a^2 = 44 \log 3$

$\therefore \quad a = \pm \sqrt{3}$

9 We know that,

$A_1 + A_2 + \ldots + A_n = nA$, where

$A = \frac{a+b}{2}$

$\therefore \quad 364 = \left(\frac{7 + 49}{2}\right)n$

$\Rightarrow \quad n = \frac{364 \times 2}{56} = 13$

10 Given, $\quad x = \frac{1}{9}(999\ldots9) = \frac{1}{9}(10^{20} - 1)$

$y = \frac{1}{3}(999\ldots9) = \frac{1}{3}(10^{10} - 1)$

and $\quad z = \frac{2}{9}(999\ldots9) = \frac{2}{9}(10^{10} - 1)$

$\therefore \quad \frac{x - y^2}{z} = \frac{10^{20} - 1 - (10^{10} - 1)^2}{2(10^{10} - 1)}$

$= \frac{10^{10} + 1 - (10^{10} - 1)}{2} = 1$

11 Let a be the first term and d be the common difference.

Then, we have $a + d, a + 4d, a + 8d$ in GP,

i.e. $\quad (a + 4d)^2 = (a + d)(a + 8d)$

$\Rightarrow a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$

$\Rightarrow \quad 8d^2 = ad$

$\Rightarrow \quad 8d = a \quad [\because d \neq 0]$

Now, common ratio,

$r = \frac{a + 4d}{a + d} = \frac{8d + 4d}{8d + d} = \frac{12d}{9d} = \frac{4}{3}$

12 Let a, ar, ar^2 be in GP (where, $r > 1$).

On multiplying middle term by 2, we get that $a, 2ar, ar^2$ are in AP.

$\Rightarrow 4ar = a + ar^2 \Rightarrow r^2 - 4r + 1 = 0$

$\Rightarrow \quad r = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$

$\therefore \quad r = 2 + \sqrt{3} \quad [\because \text{AP is increasing}]$

13 Let the GP be $a, ar, ar^2, ar^3, ar^{2n-2}, ar^{2n-1}$.

where, $a, ar^2, ar^4, ar^6, \ldots$ occupy odd places and $ar, ar^3, ar^5, ar^7, \ldots$ occupy even places.

Given, sum of all terms $= 5 \times$ sum of terms occupying odd places, i.e.

$a + ar + ar^2 + \ldots + ar^{2n-1}$

$= 5 \times (a + ar^2 + ar^4 + \ldots + ar^{2n-2})$

$\Rightarrow \quad \frac{a(r^{2n} - 1)}{r - 1} = \frac{5a[(r^2)^n - 1]}{r^2 - 1}$

$\left[\because S_n = \frac{a(r^n - 1)}{r - 1}\right]$

$\Rightarrow \quad \frac{r^{2n} - 1}{r - 1} = \frac{5(r^{2n} - 1)}{(r-1)(r+1)}$

$\Rightarrow \quad 1 = \frac{5}{r + 1} \Rightarrow r + 1 = 5 \Rightarrow r = 4$

14 Let $S = 0.7 + 0.77 + 0.777 + \ldots$ upto 20 terms

$= \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \ldots$ upto 20 terms

$= 7\left[\frac{1}{10} + \frac{11}{10^2} + \frac{111}{10^3} + \ldots \text{ upto 20 terms}\right]$

$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ upto 20 terms}\right]$

$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots \text{ upto 20 terms}\right]$

$= \frac{7}{9}\left[(1 + 1 + \ldots \text{ upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots \text{ upto 20 terms}\right)\right]$

$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}}{1 - \frac{1}{10}}\right]$

$\left[\because \text{sum of n terms of GP, } S_n = \frac{a(1 - r^n)}{1 - r}, \text{ where } r < 1\right]$

$= \frac{7}{9}\left[20 - \frac{1}{9}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}\right]$

$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + 10^{-20}]$

15 x, y, z are in GP

$\Leftrightarrow \quad y^2 = xz$

$\Leftrightarrow \quad x$ is factor of y. Which is not possible, as y is a prime number.

If $x = 3$, $y = 5$ and $z = 7$, then they are in AP.

Thus, x, y and z may be in AP but not in GP.

16 Clearly,

$$1+n+n^2+\ldots+n^{127}=\frac{n^{128}-1}{n-1}\qquad\left[\because S_n=\frac{a(r^n-1)}{r-1}\right]$$

$$=\frac{(n^{64}-1)(n^{64}+1)}{n-1}$$

$$=(1+n+n^2+\ldots+n^{63})(n^{64}+1)$$

Thus, the largest value of m for which n^m+1 divides $1+n+n^2+\ldots+n^{127}$ is 64.

17 Since, $S_\infty=\dfrac{x}{1-r}=5\Rightarrow r=\dfrac{5-x}{5}$

For infinite GP, $|r|<1$

$\Rightarrow -1<\dfrac{5-x}{5}<1\Rightarrow -10<-x<0$

$\therefore\quad 0<x<10$

18 Sum of the area of the squares which carried upto infinity

$$=a^2+\frac{a^2}{2}+\frac{a^2}{4}+\ldots$$

$$=\frac{a^2}{1-\frac{1}{2}}=2a^2m^2$$

19 Clearly, $1+(1+a)b+(1+a+a^2)b^2+(1+a+a^2+a^3)b^3+\ldots\infty$

$$=\sum_{n=1}^{\infty}(1+a+a^2+\ldots+a^{n-1})b^{n-1}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1-a^n}{1-a}\right)b^{n-1}$$

$$=\frac{1}{1-a}\left[\sum_{n=1}^{\infty}b^{n-1}-\sum_{n=1}^{\infty}a^nb^{n-1}\right]$$

$$=\frac{1}{1-a}\left[\sum_{n=1}^{\infty}b^{n-1}-a\sum_{n=1}^{\infty}(ab)^{n-1}\right]$$

$$=\frac{1}{1-a}[1+b+b^2+\ldots\infty]-\frac{a}{1-a}[1+ab+(ab)^2+\ldots]$$

$$=\frac{1}{1-a}\cdot\frac{1}{1-b}-\frac{a}{1-a}\cdot\frac{1}{1-ab}$$

$[\because |b|<1 \text{ and } |ab|=|a||b|<1]$

$$=\frac{1-ab-a(1-b)}{(1-a)(1-b)(1-ab)}$$

$$=\frac{1-ab-a+ab}{(1-a)(1-b)(1-ab)}$$

$$=\frac{1}{(1-b)(1-ab)}$$

20 Let the time taken to save ₹ 11040 be $(n+3)$ months.
For first 3 months he saves ₹ 200 each month.

In $(n+3)$ months, $3\times200+\dfrac{n}{2}\{2(240)+(n-1)\times40\}=11040$

$\Rightarrow\quad 600+\dfrac{n}{2}\{40(12+n-1)\}=11040$

$\Rightarrow\quad 600+20n(n+11)=11040$

$\Rightarrow\quad 30+n^2+11n=552$

$\Rightarrow\quad n^2+11n-522=0$

$\Rightarrow\quad n^2+29n-18n-522=0$

$\Rightarrow\quad n(n+29)-18(n+29)=0$

$\Rightarrow\quad (n-18)(n+29)=0$

$\therefore\quad n=18$ [neglecting $n=-29$]

$\therefore$ Total time $=(n+3)=21$ months

21 Since, $g=\sqrt{ab}$. Also, a,p,q and b are in AP.

So, common difference d is $\dfrac{b-a}{3}$.

$\therefore\quad p=a+d=a+\dfrac{b-a}{3}=\dfrac{2a+b}{3}$

$q=b-d=b-\dfrac{b-a}{3}=\dfrac{a+2b}{3}$

Now, $(2p-q)(p-2q)$

$$=\frac{(4a+2b-a-2b)}{3}\cdot\frac{(2a+b-2a-4b)}{3}$$

$$=-ab=-g^2$$

22 Here, $a=486$ and $b=\dfrac{2}{3}$

We know that, $G_r=a\left(\dfrac{b}{a}\right)^{\frac{r}{n+1}}$

$\therefore\ G_4=486\left(\dfrac{2}{3}\cdot\dfrac{1}{486}\right)^{4/6}$ [$\because$ here, $n=5$]

$$=486\left(\frac{1}{3\cdot243}\right)^{4/6}$$

$$=486\left(\frac{1}{729}\right)^{4/6}=486\cdot\frac{1}{3^4}=6$$

23 Let $x=1+\dfrac{1}{50}$ and S_{50} be the sum of first 50 terms of the given series.

Then, $\quad S_{50}=1+2x+3x^2+\ldots+50x^{49}\quad\ldots$(i)

$\Rightarrow\quad xS_{50}=x+2x^2+\ldots+49x^{49}+50x^{50}\quad\ldots$(ii)

$\Rightarrow(1-x)S_{50}=1+x+x^2+x^3+\ldots+x^{49}-50x^{50}$

[subtracting Eq. (ii) from Eq. (i)]

$\Rightarrow\quad S_{50}(1-x)=\dfrac{1-x^{50}}{1-x}-50x^{50}$

$\Rightarrow\quad S_{50}\left(\dfrac{-1}{50}\right)=\dfrac{1-x^{50}}{\left(\dfrac{-1}{50}\right)}-50x^{50}\qquad\left[\because x=1+\dfrac{1}{50}\right]$

$\Rightarrow\quad S_{50}\left(\dfrac{-1}{50}\right)=-50+50x^{50}-50x^{50}$

$\Rightarrow\quad S_{50}=2500.$

24 Given, $k\cdot10^9=10^9+2(11)^1(10)^8+3(11)^2(10)^7+\ldots+10(11)^9$

$$k=1+2\left(\frac{11}{10}\right)+3\left(\frac{11}{10}\right)^2+\ldots+10\left(\frac{11}{10}\right)^9\quad\ldots\text{(i)}$$

$$\left(\frac{11}{10}\right)k=1\left(\frac{11}{10}\right)+2\left(\frac{11}{10}\right)^2+\ldots+9\left(\frac{11}{10}\right)^9+10\left(\frac{11}{10}\right)^{10}\quad\ldots\text{(ii)}$$

On subtracting Eq.(ii) from Eq.(i), we get

$$k\left(1-\frac{11}{10}\right)=1+\frac{11}{10}+\left(\frac{11}{10}\right)^2+\ldots+9\left(\frac{11}{10}\right)^9-10\left(\frac{11}{10}\right)^{10}$$

$$\Rightarrow k\left(\frac{10-11}{10}\right)=\frac{1\left[\left(\frac{11}{10}\right)^{10}-1\right]}{\left(\frac{11}{10}-1\right)}-10\left(\frac{11}{10}\right)^{10}$$

$\left[\because \text{in GP, sum of } n \text{ terms}=\dfrac{a(r^n-1)}{r-1},\ \text{when } r>1\right]$

$$\Rightarrow -k=10\left[10\left(\frac{11}{10}\right)^{10}-10-10\left(\frac{11}{10}\right)^{10}\right]$$

$\therefore\quad k=100$

25 Let $S=1+\dfrac{2}{3}+\dfrac{6}{3^2}+\dfrac{10}{3^3}+\dfrac{14}{3^4}+\ldots$

$$=1+\frac{2}{3}\left[1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\ldots\right]$$

$$=1+\frac{2}{3}\left[\frac{1}{1-1/3}+\frac{2\cdot1/3}{(1-1/3)^2}\right]$$

$\left[\because \text{sum of infinite AGP, is } S_\infty=\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}\right]$

$$=1+\frac{2}{3}\left[\frac{3}{2}+\frac{2}{3}\cdot\frac{9}{4}\right]=1+\frac{2}{3}\cdot2\cdot\frac{3}{2}=3$$

26 $1^3+3^3+\ldots+39^3=1^3+2^3+3^3+\ldots+40^3-(2^3+4^3+6^3+\ldots+40^3)$

$$=\left(\frac{40\times41}{2}\right)^2-8(1^3+2^3+3^3+\ldots+20^3)$$

$$=(20\times41)^2-8\left(\frac{20\times21}{2}\right)^2$$

$$=20^2[41^2-2(21)^2]$$

$$=319600$$

27 Let $a_1 = a$ and d = common difference

$\because a_1 + a_5 + a_9 + \cdots + a_{49} = 416$

$\therefore \quad a + (a + 4d) + (a + 8d) + \ldots (a + 48d) = 416$

$\Rightarrow \quad \frac{13}{2}(2a + 48d) = 416$

$\Rightarrow \quad a + 24d = 32 \quad \ldots$(i)

Also, we have $a_9 + a_{43} = 66$

$\therefore a + 8d + a + 42d = 66$

$\Rightarrow \quad 2a + 50d = 66$

$\Rightarrow \quad a + 25d = 33 \quad \ldots$(ii)

Solving Eqs. (i) and (ii), we get

$a = 8$ and $d = 1$

Now, $a_1^2 + a_2^2 + a_3^2 + \cdots + a_{17}^2 = 140m$

$8^2 + 9^2 + 10^2 + \ldots + 24^2 = 140m$

$\Rightarrow (1^2 + 2^2 + 3^2 + \ldots + 24^2) - (1^2 + 2^2 + 3^2 + \ldots + 7^2) = 140m$

$\Rightarrow \quad \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$

$\Rightarrow \quad \frac{3 \times 7 \times 8 \times 5}{6}(7 \times 5 - 1) = 140m$

$\Rightarrow \quad 7 \times 4 \times 5 \times 34 = 140m$

$\Rightarrow \quad 140 \times 34 = 140m$

$\Rightarrow \quad m = 34$

28 We have, $f(x) = ax^2 + bx + c$

Now, $f(x + y) = f(x) + f(y) + xy$

Put $y = 0 \Rightarrow f(x) = f(x) + f(0) + 0$

$\Rightarrow \quad f(0) = 0$

$\Rightarrow \quad c = 0$

Again, put $y = -x$

$\therefore \quad f(0) = f(x) + f(-x) - x^2$

$\Rightarrow \quad 0 = ax^2 + bx + ax^2 - bx - x^2$

$\Rightarrow \quad 2ax^2 - x^2 = 0 \Rightarrow a = \frac{1}{2}$

Also, $a + b + c = 3$

$\Rightarrow \quad \frac{1}{2} + b + 0 = 3 \Rightarrow b = \frac{5}{2}$

$\therefore \quad f(x) = \frac{x^2 + 5x}{2}$

Now, $\quad f(n) = \frac{n^2 + 5n}{2} = \frac{1}{2}n^2 + \frac{5}{2}n$

$\therefore \quad \sum_{n=1}^{10} f(n) = \frac{1}{2}\sum_{n=1}^{10} n^2 + \frac{5}{2}\sum_{n=1}^{10} n$

$= \frac{1}{2} \cdot \frac{10 \times 11 \times 21}{6} + \frac{5}{2} \times \frac{10 \times 11}{2}$

$= \frac{385}{2} + \frac{275}{2} = \frac{660}{2} = 330$

29 Series $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$

$= 4\{1 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + \ldots\}$

$\therefore \quad T_n = 4n \cdot n^2$

and $S_n = \Sigma T_n = 4\Sigma n^3 = 4\left[\frac{n(n+1)}{2}\right]^2$

Now, $S_{10} = [10 \cdot (10 + 1)]^2$

$= (110)^2 = 12100$

30 We have,

$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$

A = sum of first 20 terms

B = sum of first 40 terms

$\therefore \quad A = 1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots + 2 \cdot 20^2$

$A = (1^2 + 2^2 + 3^2 + \ldots + 20^2) + (2^2 + 4^2 + 6^2 + \ldots + 20^2)$

$A = (1^2 + 2^2 + 3^2 + \ldots + 20^2) + 4(1^2 + 2^2 + 3^2 + \ldots + 10^2)$

$A = \frac{20 \times 21 \times 41}{6} + \frac{4 \times 10 \times 11 \times 21}{6}$

$A = \frac{20 \times 21}{6}(41 + 22) = \frac{20 \times 21 \times 63}{6}$

Similarly,

$B = (1^2 + 2^2 + 3^2 + \ldots + 40^2) + 4(1^2 + 2^2 + \ldots + 20^2)$

$B = \frac{40 \times 41 \times 81}{6} + \frac{4 \times 20 \times 21 \times 41}{6}$

$B = \frac{40 \times 41}{6}(81 + 42) = \frac{40 \times 41 \times 123}{6}$

Now, $B - 2A = 100\lambda$

$\therefore \quad \frac{40 \times 41 \times 123}{6} - \frac{2 \times 20 \times 21 \times 63}{6} = 100\lambda$

$\Rightarrow \quad \frac{40}{6}(5043 - 1323) = 100\lambda$

$\Rightarrow \quad \frac{40}{6} \times 3720 = 100\lambda$

$\Rightarrow \quad 40 \times 620 = 100\lambda$

$\Rightarrow \quad \lambda = \frac{40 \times 620}{100} = 248$

31 Write the nth term of the given series and simplify it to get its lowest form.Then, apply, $S_n = \Sigma T_n$.

Given series is

$\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots \infty$

Let T_n be the nth term of the given series.

$\therefore T_n = \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots \text{ upto } n \text{ terms}}$

$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{n^2} = \frac{(n+1)^2}{4}$

Now, $S_9 = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}[(2^2 + 3^2 + \ldots + 10^2) + 1^2 - 1^2]$

$= \frac{1}{4}\left[\frac{10(10+1)(20+1)}{6} - 1\right]$

$= \frac{384}{4} = 96$

32 $T_n = \frac{2n + 1}{(1^2 + 2^2 + \ldots + n^2)}$

$= \frac{2n + 1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$

$= 6\left(\frac{1}{n} - \frac{1}{(n+1)}\right)$

$T_1 = 6\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 6\left[\frac{1}{2} - \frac{1}{3}\right], \ldots$

$T_{11} = 6\left[\frac{1}{11} - \frac{1}{12}\right]$

$\therefore S = 6\left[\frac{1}{1} - \frac{1}{12}\right] = \frac{6 \times 11}{12} = \frac{11}{2}$

33 nth term of the series is

$T_n = \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)}$

$\Rightarrow T_n = 2\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$

$\Rightarrow T_1 = 2\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right),$

$T_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right), \ldots, T_{10} = 2\left(\frac{1}{10} - \frac{1}{11}\right)$

$\therefore S_{10} = T_1 + T_2 + \ldots + T_{20}$

$= 2\left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{10} - \frac{1}{11}\right]$

$= 2\left(1 - \frac{1}{11}\right)$

$= 2 \cdot \frac{10}{11} = \frac{20}{11}$

SESSION 2

1 Let $S = 1^2 + 3^2 + 5^2 + \ldots + 25^2$

$= (1^2 + 2^2 + 3^2 + 4^2 + \ldots + 25^2) - (2^2 + 4^2 + 6^2 + \ldots + 24^2)$

$= (1^2 + 2^2 + 3^2 + 4^2 + \ldots + 25^2) - 2^2(1 + 2^2 + 3^2 + \ldots + 12^2)$

$= \frac{25(25+1)(2 \times 25 + 1)}{6} - 4 \times \frac{12(12+1)(2 \times 12 + 1)}{6}$

$= \frac{25 \times 26 \times 51}{6} - \frac{4 \times 12 \times 13 \times 25}{6}$

$= 25 \times 13 \times 17 - 4 \times 2 \times 13 \times 25$

$= 5525 - 2600 = 2925$

2 Now, $f(2) = f(1 + 1)$

$= f(1) \cdot f(1) = 2^2$ and $f(3) = 2^3$

Similarly, $\quad f(n) = 2^n$

$\therefore \; 16(2^n - 1) = \sum_{r=1}^{n} f(a + r) = \sum_{r=1}^{n} 2^{a+r}$

$= 2^a(2 + 2^2 + \ldots + 2^n)$

$$= 2^a \cdot 2\left(\frac{2^n - 1}{2 - 1}\right) \qquad \text{[GP series]}$$

$$= 2^{a+1}(2^n - 1)$$

$\Rightarrow \quad 2^{a+1} = 16 = 2^4$

$\therefore \quad a = 3$

3 Let GP be $a, ar, ar^2, \dots, |r| < 1$.

According to the question,

$$\frac{a}{1-r} = \frac{7}{2}, \frac{a^2}{1-r^2} = \frac{147}{16}$$

On eliminating a, we get

$$\frac{147}{16}(1-r^2) = \left(\frac{7}{2}\right)^2 (1-r)^2$$

$\Rightarrow 3(1+r) = 4(1-r) \Rightarrow r = \frac{1}{7}, a = 3$

$\therefore$ Sum of cubes

$$= \frac{a^3}{1-r^3} = \frac{(3)^3}{1-\left(\frac{1}{7}\right)^3} = \frac{1029}{38}$$

4 Let $S = \frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \dots$ ...(i)

and $\frac{S}{13} = \frac{5}{13^2} + \frac{55}{13^3} + \dots$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{12}{13}S = \frac{5}{13} + \frac{50}{13^2} + \frac{500}{13^3} + \dots$$

which is a GP with common ratio $\frac{10}{13}$.

$$\therefore \quad S = \frac{13}{12} \times \left[\frac{5}{13} \div \left(1 - \frac{10}{13}\right)\right] = \frac{65}{36}$$

$$\left[\because S_\infty = \frac{a}{1-r}\right]$$

5 Here, $T_1 = S_1 = 2(1) + 3(1)^2 = 5$

$T_2 = S_2 - S_1 = 16 - 5 = 11$

$[\because S_2 = 2(2) + 3(2)^2 = 16]$

$T_3 = S_3 - S_2 = 33 - 16 = 17$

$[\because S_3 = 2(3) + 3(3)^2 = 33]$

Hence, sequence is 5, 11, 17.

$\therefore \quad a = 5$ and $d = 6$

For new AP, $A = 5$, $D = 2 \times 6 = 12$

$$\therefore \quad S'_n = \frac{n}{2}[2 \times 5 + (n-1)12]$$

$$= 6n^2 - n$$

6 Sum of three infinite GP's are

$$x = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta},$$

Similarly,

$$y = \frac{1}{\cos^2\theta} \text{ and } z = \frac{1}{1-\cos^2\theta\sin^2\theta}$$

Now, $\frac{1}{x} + \frac{1}{y} = 1 \quad [\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow \quad x + y = xy$

and $\quad \frac{1}{z} = 1 - \cos^2\theta\sin^2\theta$

$$= 1 - \frac{1}{xy} = \frac{xy-1}{xy}$$

$\Rightarrow \quad xy = xyz - z$

$\therefore \quad xyz = xy + z = x + y + z$

7 Let $S = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots + \text{to } \infty$

Since, $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \text{ to } \infty = \frac{\pi^4}{90}$

$$\therefore \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \text{ to } \infty\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \text{ to } \infty\right) = \frac{\pi^4}{90}$$

$$\Rightarrow \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \text{ to } \infty\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \text{ to } \infty\right) = \frac{\pi^4}{90}$$

$$\Rightarrow S + \frac{1}{16} \cdot \frac{\pi^4}{90} = \frac{\pi^4}{90}$$

$$\left(\because \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \text{ to } \infty = \frac{\pi^4}{90}\right)$$

$$\Rightarrow \quad S = \frac{\pi^4}{90}\left(1 - \frac{1}{16}\right) = \frac{15\pi^4}{16 \times 90} = \frac{\pi^4}{96}$$

8 Let $a_n = ar^{n-1}$.

Then, $S_n = \frac{a(1-r^n)}{1-r}$

and $\quad S'_n = \dfrac{\left(\frac{1}{a}\right)\left[1 - \left(\frac{1}{r}\right)^n\right]}{1 - \frac{1}{r}}$

$$\left[\because \text{first term of } \left\{\frac{1}{a_n}\right\} \text{ is } \frac{1}{a} \text{ and common ratio is } \frac{1}{r}\right]$$

$$= \frac{\left(\frac{1}{a}\right)(r^n - 1)}{r^n(r-1)} \cdot r$$

$$= \frac{1-r^n}{1-r} \cdot \frac{1}{a \cdot r^{n-1}}$$

$$= \frac{1-r^n}{1-r} \cdot \frac{1}{a_n} = \frac{a(1-r^n)}{1-r} \cdot \frac{1}{a\,a_n}$$

$$= S_n \cdot \frac{1}{a_1 a_n}$$

$\Rightarrow \quad S_n = a_1 a_n S'_n$

9 Let S_{10} be the sum of first ten terms of the series.

Then, we have

$$S_{10} = \left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + 4^2 + \left(\frac{24}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \frac{1}{5^2}(8^2 + 12^2 + 16^2 + 20^2 + 24^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \dots + 11^2)$$

$$= \frac{16}{25}((1^2 + 2^2 + \dots + 11^2) - 1^2)$$

$$= \frac{16}{25}\left(\frac{11 \cdot (11+1)(2 \cdot 11 + 1)}{6} - 1\right)$$

$$= \frac{16}{25}(506 - 1) = \frac{16}{25} \times 505$$

$\Rightarrow \quad \frac{16}{5}m = \frac{16}{25} \times 505$

$\Rightarrow \quad m = 101$

10 Given, m is the AM of l and n.

$\therefore \quad l + n = 2m$

and G_1, G_2, G_3 are geometric means between l and n.

So, l, G_1, G_2, G_3, n are in GP.

Let r be the common ratio of this GP.

$\therefore \ G_1 = lr, G_2 = lr^2, G_3 = lr^3;$

$$n = lr^4 \Rightarrow r = \left(\frac{n}{l}\right)^{1/4}$$

Now, $G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$

$$= l^4 \times r^4(1 + 2r^4 + r^8) = l^4 \times r^4(r^4 + 1)^2$$

$$= l^4 \times \frac{n}{l}\left(\frac{n+l}{l}\right)^2 = ln \times 4m^2 = 4lm^2n$$

$[\because n + l = 2m]$

11 Given series is a geometric series with $a = \sqrt{2} + 1$ and $r = \sqrt{2} - 1$.

$\therefore$ Required sum

$$= \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}}$$

$$= \frac{(\sqrt{2}+1)(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$= \frac{2\sqrt{2} + 2 + 2 + \sqrt{2}}{4-2} = \frac{4+3\sqrt{2}}{2}$$

12 Clearly, the common terms of the given sequences are

31, 41, 51, ...

Now, 100th term of 1, 11, 21, 31, ... is

$1 + 99 \times 10 = 991$

and 100th term of 31, 36, 41, 46, ... is

$31 + 99 \times 5 = 526$.

Let the largest common term be 526.

Then, $526 = 31 + (n-1)10$

$\Rightarrow \quad (n-1)10 = 495$

$\Rightarrow \quad n - 1 = 49.5$

$\Rightarrow \quad n = 50.5$

But n is an integer, $n = 50$.

Hence, the largest common term is $31 + (50 - 1)10 = 521$.

13 Since, a, b, c are in GP.

$\therefore \quad b^2 = ac \quad \ldots(i)$

Also, as x is A between a and b

$\therefore \quad x = \frac{a+b}{2} \quad \ldots(ii)$

Similarly, $y = \frac{b+c}{2} \quad \ldots(iii)$

Now, consider $\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$

[using Eqs. (ii) and (iii)]

$$= 2\left[\frac{ab + ac + ac + bc}{ab + ac + b^2 + bc}\right]$$

$$= 2\left[\frac{ab + bc + 2ac}{ab + bc + 2ac}\right]$$

$= 2$ [using Eq. (i)]

14 Since, a, b, c are in AP

$\therefore \quad 2b = a + c \quad \ldots(i)$

Also, as a^2, b^2 and c^2 are in GP

$\therefore \quad b^4 = a^2c^2 \quad \ldots(ii)$

$\because \quad a + b + c = \frac{3}{2}$

$\therefore \quad 3b = \frac{3}{2}$ [using Eq. (i)]

$\Rightarrow \quad b = \frac{1}{2}$

$\Rightarrow \quad a + c = 1$ [using Eq. (i)]

and $ac = \frac{1}{4}$ or $-\frac{1}{4}$ [using Eq. (ii)]

Case I When $a + c = 1$ and $ac = \frac{1}{4}$

In this case,

$(a - c)^2 = (a + c)^2 - 4ac = 0$

$\Rightarrow \quad a = c$

But $a \neq c$, as $a < c$.

Case II When $a + c = 1$ and $ac = -\frac{1}{4}$

In this case, $(a - c)^2 = 1 + 1 = 2$

$\Rightarrow \quad a - c = \pm\sqrt{2}$

But $a < c$, $a - c = -\sqrt{2}$

On solving $a + c = 1$

and $a - c = -\sqrt{2}$, we get

$$a = \frac{1}{2} - \frac{1}{\sqrt{2}}.$$

15 We have, $225a^2 + 9b^2 + 25c^2 - 75ac - 45ab - 15bc = 0$

$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$\Rightarrow \frac{1}{2}[(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2] = 0$

$\Rightarrow \quad 15a = 3b, 3b = 5c$ and $5c = 15a$

$\therefore \quad 15a = 3b = 5c$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{5} = \frac{c}{3} = \lambda$ (say)

$\Rightarrow \quad a = \lambda, b = 5\lambda, c = 3\lambda$

Hence, b, c and a are in AP.

16 Here, S_r is sum of an infinite GP, r is first term and $\frac{1}{r+1}$ is common ratio

$$S_r = \frac{r}{1 - \frac{1}{r+1}} = r + 1$$

$$\Rightarrow \sum_{r=1}^{10} S_r^2 = 2^2 + 3^2 + \ldots + 11^2$$

$$= 1^2 + 2^2 + 3^2 + \ldots + 11^2 - 1$$

$$= \frac{11 \times 12 \times 23}{6} - 1 = 505$$

17 Statement I

Let $S = (1) + (1+2+4) + (4+6+9) + (9+12+16) + \ldots + (361+380+400)$

$= (0+0+1) + (1+2+4) + (4+6+9) + (9+12+16) + \ldots + (361+380+400)$

Now, we can clearly observe the elements in each bracket.

The general term of the series is

$T_r = (r-1)^2 + (r-1)r + (r^2)$

Now, the sum to n terms of the series is

$$S_n = \sum_{r=1}^{n}[(r-1)^2 + (r-1)r + (r)^2]$$

$$= \sum_{r=1}^{n}\left[\frac{r^3 - (r-1)^3}{r - (r-1)}\right]$$

$[\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)]$

$$= \sum_{r=1}^{n}[r^3 - (r-1)^3]$$

$$= (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + [n^3 - (n-1)^3]$$

Rearranging the terms, we get

$S_n = -0^3 + (1^3 - 1^3) + (2^3 - 2^3) + (3^3 - 3^3) + \ldots + [(n-1)^3 - (n-1)^3] + n^3$

$= n^3$

$\Rightarrow \quad S_{20} = 8000$

Hence, Statement I is correct.

Statement II We have, already proved in the Statement I that

$$S_n = \sum_{r=1}^{n}(r^3 - (r-1)^3) = n^3$$

Hence, Statement II is also correct and it is a correct explanation for Statement I.